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STRONGLY MINIMAL EXPANSIONS OF ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT

(1) We construct a strongly minimal expansion of an algebraically closed field of a given characteristic. Actually we show a much more general result, implying for example the existence of a strongly minimal set with two different field structures of distinct characteristics.

(2) A strongly minimal expansion of an algebraically closed field that preserves the algebraic closure relation must be an expansion by (algebraic) constants.

1. Introduction

The first result mentioned in the abstract should be seen as part of the attempt to understand the nature of $\aleph_1$-categorical theories, following the refutation of Zil'ber's conjecture [H]. Specifically it refutes a conjecture made by Poizat as part of a program to classify simple groups of finite Morley rank avoiding the full Zil'ber conjecture. [P1]. It seems that more experience with such constructions will have to be gained before a general program can be reformulated.

The second result was obtained several years ago as a lemma in the classification of the geometries of strongly minimal sets of Zil'ber type (yet unpublished). It is closely related to the theorem of [M], which was obtained independently, by a somewhat different method. It seemed appropriate to include it in the present paper because of its similarity to the definable multiplicity property considered in section 2. For fields, in dimension 1, the DMP states that curves remain

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strongly minimal under specialization of the parameters. The result (2) is that they remain so in strongly minimal expansions of the structure. However, the definable multiplicity property has an elementary proof, whereas (2) appears to require some use of Riemann–Roch.

All structures considered in this paper will be assumed to have a countable language. In section 4 we will show that any two strongly minimal sets with the definable multiplicity property can be amalgamated to a single strongly minimal set. This provides many examples of strongly minimal sets, not of classical type, with interesting geometries. (Unlike the "flat" geometry of the structure constructed in [H].) The geometry can be seen as "relatively flat" over the geometries of the given strongly minimal sets, however. In particular it can be shown that if G is a connected group definable in the strongly minimal amalgam of D_1, D_2 then there exist connected groups G_1, G_2 definable over D_1, D_2 respectively, and a definable surjective group homomorphism $f\colon G \to G_1 \times G_2$ with finite (central) kernel. It would be good to have the sharper result with the arrow reversed $f\colon G_1 \times G_2 \twoheadrightarrow G$). This would require a closer analysis.

It seems likely that the construction of Theorem 1 can be generalized to prove the following. Suppose T_1, T_2 are strongly minimal theories with the DMP in languages $L_1, L_2, L_1 \cap L_2 = L_0$, $T_1 \mid L_0 = T_2 \mid L_0 = T_0$, and T_0 is the theory of infinite vector spaces over a finite field F. Then there exists a strongly minimal extension of $T_1 \cup T_2$. Here is a simple example showing that this is false for arbitrary T_0. Let T_0 be the theory of divisible torsion free Abelian groups with two distinguished elements c, d independent over the rationals. Let $L_i = L \cup \{\alpha_i\}$, $T_i \vdash$ "α_i is an endomorphism, $\alpha_i(c) = d$", $T_1 \vdash$ "$\alpha_1^2 + 1 = 0$", $T_2 \vdash$ "$\alpha_2^2 + 1$ is surjective." The kernel of $\alpha_1 - \alpha_2$ is infinite and of infinite index in any model of $T_1 \cup T_2$.

Another example: T_i = theory of free action of G_i on an infinite set. $G_1 = (\mathbb{Z}/2\mathbb{Z})^2$, $G_2 = (\mathbb{Z}/4\mathbb{Z})$, T_0 = theory of an equivalence relation with 4 element classes. In this example, a strongly minimal set of T_0 fails to remain strongly minimal in both T_1 and T_2.

A final variation that is likely quite accessible at this point is the construction of a strongly minimal set supporting a field structure in dimension 2 but not in dimension 1 (conjecture of Berline–Lascar). Such a structure would presumably be "flat" in all odd dimensions, but not in the even ones.

For notation and basic results the reader is referred to [Pi].

2. The definable multiplicity property

Definition: A strongly minimal set D has the **definable multiplicity property** (DM) if whenever $\varphi(\bar{x}, \bar{a})$ defines a subset of D^n of rank k, multiplicity m, then for some $\psi \in tp(\bar{a}/\emptyset)$, for all $\bar{a}'$, if $\psi(\bar{a}')$ then $\varphi(\bar{x}, \bar{a}')$ has rank k, multiplicity m. ∎

An equivalent statement: Let $f\colon U \to V$ be a definable map between definable sets. Then for any $k, m, \{v \in V : f^{-1}(v)$ has rank k, multiplicity $m\}$ is a definable subset of V. ∎

Other equivalent statements can be formulated in the language of reduction of structure groups of finite coverings.

Remarks:

(1) The corresponding property for rank alone is true in any $\aleph_1$-categorical theory [B].

(2) If D is disintegrated, then D has DM.

(3) If D is locally modular, then for some definable equivalence relation E with finite classes, D/E has the DM.

(4) A locally modular D need not have the DM: let V be a vector space over the rationals, with distinguished element a_0. Let $D = V \times \{0,1\}$, $\pi\colon D \to V$ the projection, and define $f\colon D \to D$ by $f(v,i) = (v + a, i)$. (D, V, π, f) is strongly minimal. Let $C(v) = \{(d_1, d_2)\colon\ \pi(d_1) - \pi(d_2) = v\}$. For generic v, $C(v)$ is strongly minimal; but $C(na_0)$ has multiplicity 2 for all $n \in \mathbb{Z}$.

(5) Problem: does (3) hold for all strongly minimal sets?

(6) The definable multiplicity property holds for D totally categorical.

(7) The definable multiplicity property holds for D if it holds for an expansion by constants of D.

(8) If the definable multiplicity property holds, then it applies also to imaginary elements. ∎

Remark (7) is a consequence of the open mapping theorem; (2) and (3) will be proved following lemma 1. To see (8), use (7) to add constants to the language. Then for every definable set E of imaginaries there exists a definable subset E' of D^n (for some n) and a definable surjective finite-to-one map $f\colon E' \to E$. We may choose E' to have the same rank and multiplicity as E, and then the result for E follows from the one for E'.

LEMMA 1: *Let D be strongly minimal. Assume that whenever $\varphi(\bar{x},\bar{a})$ defines a strongly minimal subset of D^n then for some $\psi \in \mathrm{tp}(\bar{a}/\emptyset)$, for all $\bar{a'}$ such that $\models \psi(\bar{a'}), \varphi(\bar{x},\bar{a'})$ is strongly minimal. Then T has the DMP.*

Proof: By induction on k and m. The case $k = 0$ always holds trivially. Assume the property holds for rank k, multiplicity 1. If $\varphi(\bar{x},\bar{a})$ has rank k, multiplicity $m > 1$, one can find $\bar{a'} \supseteq \bar{a}$ and $\varphi_1,\ldots,\varphi_m$ such that $\varphi(\bar{x},\bar{a}) \equiv (\varphi_1(\bar{x},\bar{a'})\dot{v}\ldots\dot{v}\varphi_m(\bar{x},\bar{a'}))$, and $\varphi_i(\bar{x},\bar{a'})$ has rank k, multiplicity 1. The definability property for φ then follows from the corresponding property for the φ_i's. Assume the property holds for ranks $< k$, where $k > 1$. Let $\varphi(\bar{x},\bar{a})$ define a subset of D^n of rank k, multiplicity 1. By Remark 7, we may expand the language by constants, so that $\mathrm{acl}(\emptyset)$ is a model. Let $b = (b_1,\ldots,b_n)$ be a generic solution of $\varphi(\bar{x},\bar{a})$. We may assume $\mathrm{rk}(b_1,\ldots,b_{k-1}/\bar{a}) = k-1, \mathrm{rk}(b/\bar{a},b_1,\ldots,b_{k-1}) = 1$. Since $\mathrm{acl}(\bar{a},b_1,\ldots,b_{k-1})$ is a model, there exists $b' \in \mathrm{acl}(\bar{a},b_1,\ldots,b_{k-1})$ such that $\mathrm{tp}(b/\bar{a},b_1,\ldots,b_{k-1},b')$ is strongly minimal, and similarly there exists $a' \in \mathrm{acl}(\bar{a})$ such that $\mathrm{tp}(b_1,\ldots,b_{k-1},b'/\bar{a},a')$ has rank $k-1$, multiplicity 1. Applying the induction hypothesis to these these types and combining the result, we find a formula $\varphi' \in tp(\bar{b},b'/\bar{a},a')$ and $\psi' \in \mathrm{tp}(\bar{a},a')$ such that whenever ψ holds of $\bar{c},c'$, then $\varphi'(\bar{x},x',\bar{c},c')$ has rank k, multiplicity 1. Quantifying existentially over b' and a', we find $\varphi'' \in \mathrm{tp}(\bar{b},\bar{a})$ and $\psi'' \in tp(\bar{a})$ such that $\psi''(\bar{c})$ implies $\varphi''(\bar{x},\bar{c})$ has rank k, multiplicity 1. Let $\psi''' \in tp(\bar{a})$ be a formula such that whenever $\psi'''(\bar{c})$ holds, then the symmetric difference of $\varphi(\bar{x},\bar{c})$ and $\varphi''(\bar{x},\bar{c})$ has rank $< k$. Then $\psi = \psi''\&\psi'''$ shows that the property holds. ∎

Proof of Remarks 2,3: Let D be disintegrated. The lemma holds trivially for strongly minimal sets of the form: $\{(x_1,\ldots,x_n)\colon x_i = a_i \text{ for } i \in S\}$ where S is a subset of $\{1,\ldots,n\}$ of size $n-1$ (i.e. parallel to one of the axes.) But every strongly minimal subset of D^n differs from such a set, or from a 0–definable set, by a finite set. This proves (2).

Next suppose D is a strongly minimal group (with extra structure.) Again the lemma holds trivially for cosets of $\mathrm{acl}(\emptyset)$ -definable subgroups of D^n. But by [HP] every strongly minimal subset of D^n differs from such a coset by a finite set. Now by [H2], if D is locally modular, not disintegrated, then D interprets a strongly minimal group A. Moreover there exists a 0–definable equivalence relation E with finite classes on D, such that D/E is A-internal. It follows that D/E has the definable multiplicity property (using Remark 8). ∎

LEMMA 2: *Let D be strongly minimal. Whenever $\varphi(\bar{x},\bar{a})$ defines a subset of D^n of rank k, multiplicity m, then for some $\psi \in tp(\bar{a}/\emptyset)$ and $M \geq m$, for all $\bar{a'}$, if $\psi(\bar{a'})$ then $\varphi(\bar{x},a')$ has rank k, multiplicity $\leq M$.*

Proof: As in lemma 1 one reduces to the case $m = 1$. Let $\bar{b}$ be a generic solution of $\varphi(\bar{x},\bar{a})$. Reordering the indices, we may assume that $b_1,\dots,b_k$ are independent generics over $\bar{a}$. It follows that for some $m \geq 1$, the formula: "for generic $y_1,\dots,y_k$, there are exactly M $(n-k)$-tuples $y_{k+1},\dots,y_n$ such that $\varphi(\bar{x},\bar{y})$" is true of $\bar{a}$. This formula works for ψ. ∎

LEMMA 3: *Let K be an algebraically closed field. Then K has the definable multiplicity property.*

Proof: Let k be the ground field. By lemma 1 we may assume $\varphi(\bar{x},\bar{a})$ is strongly minimal. Let $\bar{b} = (b,b_1,\dots,b_n)$ be a generic solution of $\varphi(x,\bar{a})$. We may assume $b \notin \mathrm{acl}(\bar{a})$, so that $b_1,\dots,b_n \in \mathrm{acl}(\bar{a},b)$. Let $k' = \mathrm{dcl}(\bar{a},b) = \cup_n k(\bar{a},b)^{1/p^n}$. By the theorem of the primitive element, there exists c such that $k'(\bar{b}) = k'(c)$. So $\mathrm{dcl}(\bar{a},\bar{b}) = \mathrm{dcl}(\bar{a},b,c)$. Let $\psi \in \mathrm{tp}(bc/\bar{a})$ be a strongly minimal formula. Then there exists a $\bar{b}$-definable bijection between the solutions of $\psi(x_1x_2,\bar{a})$ and those of $\varphi(\bar{x},\bar{a})$ (perhaps with finitely many exceptions). Hence it suffices to prove the definable multiplicity property for ψ. ψ may be taken to have the form: $Q(x_1,x_2,\bar{b}) = 0$, where Q is a polynomial. To say that $\psi(\bar{x},\bar{b})$ is strongly minimal is to say that $Q(x_1,x_2,\bar{b})$ is absolutely irreducible, i.e. there are no non-constant polynomials $P_1(x_1,x_2)$, $P_2(x_1,x_2)$ (with coefficients in K) such that $Q = P_1P_2$. Clearly it suffices to consider polynomials whose total degree is less than that of Q. The statement that no such polynomials exist is a first order statement, proving the lemma. ∎

Remark: Let k be algebraically closed, $k[X] = k[X_1,\dots,X_n]$ the polynomial ring. If $f_1,\dots,f_m \in k[X]$, let $I(\bar{f})$ be the ideal generated by $\{f_1,\dots,f_m\}$. The definable multiplicity property for algebraically closed fields is equivalent to the following statement: if $I(\bar{f})$ is not a prime ideal, then there exist $g_1,g_2 \in k[X]$, if $g_1,g_2 \in I(\bar{f})$, $g_1 \notin I(\bar{f})$, $g_2 \notin I(\bar{f})$, with the total degree of g_1,g_2 bounded in terms of n and the degrees of the f's. This was proved in [S], and model theoretically in [D]. ∎

3. Geometry-preserving expansions of fields

LEMMA 1: *Let $\bar{M}$ be an expansion of $M, \bar{M}$ $\aleph_0$-saturated. Assume M has definable operations $+, \cdot$ making it into a field. If every unary function definable in $\bar{M}$ (with parameters) is definable in M, then every relation definable in $\bar{M}$ is parametrically definable in M.*

Proof: The word "definable" will always allow the use of parameters. By induction on n, we will show:

(a_n) Every definable relation on $\bar{M}^n$ is definable in M.

(b_n) Every definable partial function f: $\bar{M}^n \to M$ is definable in M.

For $n = 1$ (b) is hypothesized, and (a) follows. It follows from (a_1) that $\bar{M}$ is strongly minimal.

(a_{n+1}) Let $R \subseteq \bar{M}^{n+1}$ be definable. Let S be the projection of R to $\bar{M}^n$, and let $S_\infty = \{\bar{x} \in \bar{M}^n$: there are infinitely many $y \in \bar{M}$ with $(\bar{x}, y) \in R\}$. Strongly minimal sets eliminate the quantifier "there exist infinitely many", so S_∞ is definable in $\bar{M}$, hence in M. Let

$$R' = (S_\infty \times M) - R$$
$$R'' = R - (S_\infty \times M)$$

Then $R = R'' \cup ((S_\infty \times M) - R')$ is definable from S_∞, R', R''. By strong minimality, for all $\bar{x}$ there are (at most) finitely many y with $(\bar{x}, y) \in R'$, and similarly for R''. Thus we may assume that this holds for R. In other words, letting $S_m = \{\bar{x} \in M^n$: there are exactly m y such that $(\bar{x}, y) \in R\}$, we have $M^n = S_0 \cup \ldots \cup S_M$ for some M. It suffices to show that $R \cap (S_m \times M)$ is M-definable for each m.

For $\bar{a} \in S_m$, let $b_1, \ldots, b_m$ be the set of all y with $(\bar{a}, y) \in R$, and let $F_{\bar{a}}(\bar{x})$ be the polynomial $\prod(X - b_i) = \Sigma_{i \leq n} c_i X^i$. Let $c_i(\bar{a}) = c_i$. Clearly c_i is a definable function on S_m. By (b_n), c_i is definable in M. But then $R \cap (S_m \times D) = \{(\bar{a}, b)$: $\bar{a} \in S_m$ and $\Sigma c_i(\bar{a}) b^i = 0\}$ is definable in M also. This proves $(a)_{n+1}$.

(b_{n+1}) Let f: $\bar{M}^{n+1} \to \bar{M}$ be a definable partial function in $\bar{M}$. Let $a \in \bar{M}$ be a generic element, and let f_a: $\bar{M}^n \to \bar{M}$ be defined by $f_a(\bar{x}) = f(a, \bar{x})$. By ($b_n$), there exists an M- definable function g coinciding with f_a; we may write $g(\bar{x}) = G(\bar{c}, \bar{x})$ for some parameter $\bar{c} \in M^k$, so that G is defined in M without parameters. By the strong minimality of $\bar{M}$, for all but finitely many a' there exists $\bar{c'} \in M^k$ such that $f_{a'}(\bar{x}) = G(\bar{c'}, \bar{x})$. Let F be the finite set of exceptions. Clearly f is definable from $f \cap (\bar{M}^n \times (M - F))$ together with $f_c (c \in F)$ and by

(b_n) each f_c is definable in M, so we may assume $F = \emptyset$.

Define an equivalence relation $\sim$ on M^k by: $\bar{c} \sim \bar{c'}$ iff $(\forall \bar{x})(G(\bar{c}, \bar{x}) = G(\bar{c'}, \bar{x}))$. Then $\sim$ is definable in M^k. By elimination of imaginaries in M [P2], there exists an M-definable function h: $M^k \to M^j$ for some j such that $\bar{c} \sim \bar{c'}$ iff $h(\bar{c}) = h(\bar{c'})$. Define $G'(h(\bar{c}), \bar{x}) = G(\bar{c}, \bar{x})$. Then for each $a \in M$ there exists a unique $\bar{d} \in M^j$ such that $f_a(\bar{x}) = G'(\bar{d}, \bar{x})$ for all $\bar{x}$. So this $\bar{d}$ has the form $(d_1(a), \ldots, d_j(a))$ where the d_i's are definable functions in $\bar{M}$. By (b_1), they are also definable in M. Hence f is also definable in M, by $f(y, \bar{x}) = G'(d_1(y), \ldots, d_j(y), \bar{x})$. This finishes the proof. ∎

THEOREM 1: *Let F be a strongly minimal expansion of an algebraically closed field K. Let k_0 be the set of algebraic elements of F. Assume that the algebraic closure relations in K and in F coincide (over k_0). Then every definable relation of F is definable in the field language from parameters from k_0.*

Proof: We may assume F is $\aleph_0$-saturated. We state the hypothesis explicitly: for all $x_1, \ldots, x_n, y \in F$, if y is algebraic over $x_1, \ldots, x_n$ model-theoretically, then in fact y is the root of a non-zero polynomial from $k_0(x_1, \ldots, x_n)$. For the facts from algebraic geometry used in the proof, see [L].

Let L_0 be the language of fields, and let L be the possibly enriched language of F. Let $F*$ be F^{eq} as evaluated in L_0. Note first that if H is an L_0-definable subset of $F*$, and has Morley rank k in L_0, then it has Morley rank k in L. (This can be proved by induction, after co-ordinatizing H. For example if H is strongly minimal in L_0, then there exists a (parametrically) L_0-definable $R \subseteq H \times F$ such that for almost every $h \in H$ there exist finitely many (but at least one) $a \in F$ with $(h, a) \in R$. Since R is also L-definable and F is L-strongly-minimal, it follows that H has L-Morley rank 1.) The problem is that the Morley degree may (on the face of it) go up.

By lemma 1, it suffices to take a definable function f in F, and prove that it is L-definable (with parameters.) Say f is k_1-definable, $k_0 \subseteq k_1 = \mathrm{acl}(k_1)$. Consider $C = \{(x, f(x)): x \in F\}$. C is strongly minimal. Let (a, b) be a generic element of C. By assumption, there exists a nonzero polynomial $p \in k_1[X, Y]$ such that $p(a, b) = 0$. We may assume p is irreducible. Let $C' = \{(x, y): p(x, y) = 0\}$. Then C' contains C (but the degree of C' in L is unknown.) Let C'' be the set of nonsingular points of C'. Let $\tilde{C}$ be a complete nonsingular curve birationally equivalent to C', defined over k_1, and let h be a $1-1$ function, definable in the language of fields, from C'' into $\tilde{C}$. If $h[C \cap C'']$ is a cofinite subset of $\tilde{C}$ then $C \cap C''$ is cofinite in C'', so $C = C''$ (modulo a finite set), and hence C is definable in the language of fields. So suppose $h[C \cap C'']$ is not cofinite in $\tilde{C}$. If $\tilde{C}$ is a rational curve, then it is definably isomorphic to the projective line $F \cup \{\infty\}$. The image of $h[C \cap C'']$ under this isomorphism gives an infinite, co-infinite subset of $F \cup \{\infty\}$, which is impossible. So $\tilde{C}$ has genus $g \geq 1$. Let A be the Jacobian variety of $\tilde{C}$, and consider $\tilde{C}$ as a subset of A. In the language of fields $\tilde{C}$ has rank g, degree 1: a generic element of A can be written uniquely as $x_1 + \cdots + x_g$, where $(x_1, \ldots x_g)$ is a generic element of $\tilde{C} \times \cdots \times \tilde{C}$ (g times). It follows that $\tilde{C}$ still has rank g in L. However, it no longer has degree 1. For if $x_1, \ldots x_g$ are independent elenents of $h[C \cap C'']$ and $y_1, \ldots, y_g$ are independent elements of $\tilde{C} - h[C \cap C'']$, then clearly $x_1 + \cdots + x_g$, $y_1 + \cdots + y_g$ are generic but do not realize the same type. Thus A is not a connected group, so it has a definable subgroup of finite index. But in reality A does not have subgroups of finite index at all. For in the language of fields, A is a connected group, and for each $n\{a \in A\colon na = 0\}$ is finite. Thus by a rank computation $nA = A$. So every factor group of A is divisible, and hence the only finite factor group is the trivial one. This gives the required contradiction. ∎

4. Fusing two strongly minimal sets

THEOREM 2: *Let T_1, T_2 be strongly minimal theories with the definable multiplicity property, in disjoint countable languages L_1, L_2. Then there exists a strongly minimal theory T in $L_1 \cup L_2$ such that $T|L_i = T_i$. Moreover, if $D \models T$:*

(i) *L_i-definable subsets of D^n have the same rank, multiplicity in the sense of T as of T_i. T has the DMP.*

(ii) *Let $V_i \subseteq D^n$ be L_i-definable without parameters. Assume V_i avoids all diagonals $(x_\nu = x_{\nu'})$, and $\dim(V_1) + \dim(V_2) < n$. Then $V_1 \cap V_2 = \emptyset$. In particular, if $V \subseteq D^n$ is L_1-definable and L_2-definable without parameters, then V is a Boolean combination of diagonals.*

(iii) *Let $V_i \subseteq D^n$ be L_i - definable. Assume now that for every projection $\Pi : D^n \to D^m$, and all $b \in D^m$, $\dim(V_1 \cap \Pi^{-1}(b)) + \dim(V_2 \cap \Pi^{-1}(b)) \leq n - m$. Then $V_1 \cap V_2$ is finite. In particular, if D_1 is not pure equality, then D is a proper expansion of D_2.*

COROLLARY: *No strongly minimal theory in a countable language with the DMP is maximally strongly minimal.*

(This partially answers Cherlin's question whether there exist maximally strongly minimal sets, in a countable language.)

Fix T_1, T_2. Without loss of generality every formula of L_i is T_i- equivalent to an atomic formula, and L_i has no function symbols. We also assume for notational simplicity that T_i admits elimination of imaginaries. (In this connection one should keep in mind the following unpublished theorem of Lascar and Pillay: after adding constants to the language, the only sorts needed in T^{eq} are those of the form $[D^k]^n = (D^k)^n/$(action of the symmetric group on n elements.) Indeed if $acl(\emptyset)$ is infinite then $acl(e)$ is always a model, so any $e = \bar{d}/E \in D^{eq}$ has the form $\bar{d}'/E$ for some $\bar{d}' \in acl(e)$. Thus c is equi-definable with the finite set $\{\bar{d}^1, \ldots, \bar{d}^k\}$ of conjugates of $\bar{d}'$ over e).

The two principal tasks in the construction of T are the determination of theories of dimension and multiplicity. In particular if V_i is an L_i- definable subset of n-space, we must determine the dimension (according to T) of $V_1 \cap V_2$; and if this dimension is 0, the cardinality of this set. In the former we will be guided by the dimension theorem: $\dim(V_1) + \dim(V_2) + \dim(V_1 \cap V_2) = n$ should hold "in general". The dimension of V_i itself will be the same in D_i and in D. This gives a formula for $\dim(V_1 \cap V_2)$; a negative number will be interpreted as a finite intersection. If U is a projection of $V_1 \cap V_2$, and each fiber is finite for the above dimension-theoretic reasons, then $\dim(U) = \dim(V_1 \cap V_2)$. These considerations suffice to assign a dimension to each L-definable set.

DIMENSION:

Consider an $L_1 \cup L_2$-structure N such that $N|L_i$ is a submodel of a model of T_i $(i = 1, 2)$. (Equivalently, $N \models T_1^\forall \cup T_2^\forall$, the universal restrictions.)

If A is a finite subset of N, let $d_i(A)$ be the T_i-rank of A, viewed as a finite subset of a model of T. Let $d_0(A) = d_1(A) + d_2(A) - \text{card}(A)$. Let $d(A, N) = \min\{d_0(B) : A \subseteq B \subseteq N, B \text{ finite}\}$. In the rest of this section, fix N and write $d(A)$ for $d(A, N)$.

Definition: Let $A \subseteq N$ be finite, $C, B \subseteq N$ not necessarily finite.

(i) $d_0(A/B) = \lim_{B' \to B}(d_0(A \cup B') - d_0(B'))$ where the limit is taken over finite $B' \subseteq B$. It will follow from lemma 1(i) that if $A \cap B \subseteq B' \subseteq B''$ then $d_0(A \cup B'') - d_0(B'') \leq d_0(A \cup B') - d_0(B')$. (Take $A_1 = A \cup B'$, $A_2 = B''$, $A_0 =$

$B', A = A \cup B''$). Hence the limit exists.

(ii) $B \leq C$ if $B \subseteq C$ and $d_0(C'/B) \geq 0$ for all finite $C' \subseteq C$.

(iii) $d(A/B) = \min\{d(A \cup B') - d(B')\colon B' \subseteq B, B' \text{ finite }\}$. ∎

LEMMA 1: *A, B are finite subsets of N.*

(i) *If $A \subseteq N, A = A_1 \cup A_2, A_0 = A_1 \cup A_2$, then $d_0(A) \leq d_0(A_1) + d_0(A_2) - d_0(A_0)$.*

(ii) *If $A \leq B \leq N$ then $A \leq N$. This holds also if B is infinite. Assume also $\emptyset \leq N$.*

(iii) *If $A \subseteq N$, A finite, then there exists a finite $A' \leq N, A \subseteq A'$. There is a unique smallest A' with this property (call it $cl_0(A)$.) We have $d(A) = d(cl_0(A)) = d_0(cl_0(A))$.*

(iv) *if $A \subseteq B$ then $d(A) \leq d(B)$.*

(v) *$d(A/B) \leq d(A'/B)$ if $A \subseteq A'$. If $B \subseteq A$, $d(A'/B) = d(A/B) + d(A'/A)$.*

(vi) *$d(A/B) \geq d(A/B')$ if $B \subseteq B'$.*

(vii) *$d(a/B) \leq 1$ for a singleton a.*

(viii) *The relation "$d(a/B) = 0$"($a \in N, B$ a finite subset of N) defines a dependence relation on N. This means:*
Monotonicity: if $B \subseteq B'$, and $d(a/B) = 0$, then $d(a/B') = 0$.
Transitivity: if $d(a/B \cup \{c\}) = 0$, and $d(c/B) = 0$, then $d(a/B) = 0$.
Exchange: if $d(a/B \cup \{c\}) = 0$, and $d(a/B) \neq 0$, then $d(c/B \cup \{a\}) = 0$.

Proof:

(i) Clear from the corresponding inequalities for d_1 and d_2.

(ii) Let C be finite, $A \subseteq C \subseteq N$. $d_0(C/B \cap C) \geq 0$ since $B \leq N$. $d_0(B \cap C/A) \geq 0$ since $A \leq B$. So $d_0(C/A) = d_0(C/B \cap C) + d_0(B \cap C/A) \geq 0$.

(iii) Choose A' finite, $A \subseteq A' \subset N$, with $d_0(A')$ least possible. (Since $\emptyset \leq A, d_0(A')$ is bounded below by 0.) Clearly $A' \leq N$. Suppose A_1, A_2 are two distinct minimal sets containing A with this property. Let $A' = A_1 \cap A_2$. $A' \not\leq A_2$, so for some $A_2' \subseteq A_2, A_2 \subseteq A_2', d_0(A_2') < d_0(A')$. Now $d_0(A_1 \cup A_2') \leq d_0(A_1) + d_0(A_2') - d_0(A') < d_0(A_1)$. This contradicts $A_1 \leq N$.

(iv) ,(v) Clear.

(vi) Let $\bar{A} = cl_0(A \cup B), \bar{B} = cl_0(B), \bar{B}' = cl_0(B')$. Then $d(A/B') = d_0(cl_0(\bar{A} \cup \bar{B}')/B') \leq d_0(\bar{A} \cup \bar{B}'/\bar{B}') < (d_0(\bar{A}) + d_0(\bar{B}') - d_0(\bar{A} \cap \bar{B}')) - d_0(\bar{B}') = d_0(\bar{A}) - d_0(\bar{A} \cap \bar{B}') \leq d_0(\bar{A}) - d_0(\bar{B})$ (since $\bar{B} \leq \bar{A} \cap \bar{B}') = d_0(\bar{A}/\bar{B}) = d(A/B)$.

(vii) $d(\{a\} \cup B) = d_0(cl_0(\{a\} \cup B)) \leq d_0(\{a\} \cup cl_0(B)) \leq 1 + d_0(cl_0(B)) = 1 + d(B)$.

(viii) Monotonicity is given by (vi). Transitivity and the exchange property can be stated as follows: if $d(a/Bc) = 0, d(a/B) \neq 0$, then $d(c/Ba) = 0, d(c/B) \neq 0$. Evaluating $d(ac/B)$ in two ways using the second part of (v), we get $d(a/Bc) + d(c/B) = d(a/B) + d(c/Ba)$. Each of these four numbers is 0 or 1 (by (vii)), and the first and third are respectively $= 0$ and $\neq 0$, forcing the same to hold of the fourth and second. ∎

Definition: The dependence relation described in (viii) will be referred to as d-dependence. ∎

Multiplicities are assigned only in "irreducible" cases. For example we need not consider the case $V_i = V_i' \times V_i''$, where $V_1 \cap V_2 = (V_1' \cap V_2') \times (V_1'' \cap V_2'')$ and so the cardinality assigned to the pair $V_1 \cap V_2$ can be computed from those of $V_1' \cap V_2'$ and $V_1'' \cap V_2''$. We are concerned with the variation of the multiplicity with a parameter, when V_1, V_2 move in a definable family. We prepare the ground by choosing a family of representatives for the sets V_i defined in L_i with good normality and definability properties.

Definition: Let D be a strongly minimal set. A normal code for D consists of the following data:

(i) An integer m, and a formula $\psi(\bar{y_1}, \dots, \bar{y_m})$.

(ii) A definable function $f(\bar{y_1}, \dots, \bar{y_m})$.

(iii) A formula $\varphi(\bar{x}, \bar{u})$.

(iv) A formula $\theta(\bar{u})$ such that whenever $\theta(\bar{b})$ holds:

(a) $C = C(\bar{b}) =_{def} \{\bar{x}\colon \varphi(\bar{x}, \bar{b})\}$ has rank k, multiplicity 1. If $\bar{x}, \bar{y} \in C$ and $x_i = x_j$ then $y_i = y_j$.

(b) ψ is true of any m independent realizations of C.

(c) f takes the constant value $\bar{b}$ on m-tuples of realizations of C satisfying ψ.

(d) Let $\bar{x} = \bar{x}^1 {}^\frown \bar{x}^2$ be any partition of the variables $\bar{x}$ into two sets. Then for any $\bar{b}$ such that $\models \theta(\bar{b})$, if for a generic $\bar{a} \in C(\bar{b}), a = \bar{a}^1 {}^\frown \bar{a}^2, \{\bar{x}^2\colon \bar{a}^1, \bar{x}^2) \in C(\bar{b})\}$ has rank j, then for all $\bar{c}^1, \{\bar{x}^2\colon (\bar{c}^1, \bar{x}^2) \in C(\bar{b})\}$ has rank $\leq j$.

(e) If $\bar{e}^1, \dots, \bar{e}^m \in C$ and $\bar{e}$ is an element of C generic over $\bar{e}^1, \dots, \bar{e}^m$ then $\models \psi(\bar{e}, \bar{e}^1, \dots, \bar{e}^{m-1})$.

ψ is symmetric in its arguments.

ψ, f, φ, θ are assumed to have no parameters.

Write $m(c), \varphi_c, f_c, \theta_c, \psi_c, n(c), k(c)$. If $\models \theta_c(\bar{b})$, we say that $(c, \bar{b})$ is a normal code for $(C, \bar{b})$.

In an algebraically closed field, we would require C_i to be a closed set, ensuring that if two sets with a normal code agree a.e., then they are equal. In general we have to make arbitrary choices.

LEMMA 2: *Let D be a strongly minimal set D in a countable language. Assume D has the definable multiplicity property. There exists a set S of normal codes such that:*

(i) *If $c_1, c_2 \in S, n(c_1) = n(c_2), k(c_1) = k(c_2) = k$, and for some $\bar{a}_1, \bar{a}_2, \models \theta_{c_i}(\bar{a}_i)$, and the symmetric difference of $\{x\colon \varphi_{c_1}(\bar{x}, \bar{a_1})\}$ and $\{\bar{x}\colon \varphi_{c_2}(\bar{x}, \bar{a_2})\}$ has rank $< k$, then $c_1 = c_2$ (and $\bar{a_1} = \bar{a_2}$).*

(ii) *For any definable $C \subseteq D^n$ of multiplicity 1 there exists $C' \subseteq D^n$ agreeing with C up to a set of rank $< \mathrm{rk}(C)$, such that C' has a normal code $(c, \bar{b})$ with c in S.*

(iii) *If $c \in S$, and c' is obtained from c by permuting some of the variables in $\bar{x}$ then $c' \in S$.*

Definition: Choose such a set of normal codes for D_1 and for D_2, and call them the **standard** codes. ∎

Proof of lemma:

If C', C'' are two definable sets of rank k, multiplicity 1, write $C' \sim C''$ for $\mathrm{rk}(C' \triangle C'') < k$. We first explain, given a single definable set $C' \subseteq D^n$ of rank k, multiplicity 1, how to find $C \sim C'$ such that C has a normal code. First find C and $m, \varphi, f, \theta, \psi, \bar{b}$ such that (a)-(c) hold. (d) is then met by strengthening φ appropriately; this may change C but only inside its $\sim$ - class. Finally replace C by the conjunction of all formulas of the form:

$$(\forall * \bar{x}'_1)(\forall * \bar{x}'_2) \dots (\forall * \bar{x}'_f)\psi^\sigma(\bar{x}'_1, \bar{x}'_2, \dots \bar{x}'_f, \bar{x}'_{f+1} \dots, \bar{x}'_m)$$

where $(\forall * \bar{u})$ means: for a generic $\bar{u}$ such that $\varphi(\bar{u}, f(\bar{x}_1, \dots, \bar{x}_m)), f$ ranges over $0, 1, ..., m$, σ ranges over $\mathrm{Sym}(m)$, and ψ^σ has the natural meaning. It is then easy to check (using (c) for ψ) that (e) holds of the conjunction.

Next we use the countability of the language to find a family of normal codes satisfying (i) and (ii). Since (ii) is true of C iff it is true of a conjugate of C, we may assume D is countable. To choose the set of standard codes for rank k subsets of D^n, let $C_0, C_1, \dots$ be an enumeration of all rank k, multiplicity 1 definable subsets of D^n. Choose a normal code c_i for C_i inductively. Assume standard codes c_i have been chosen for C_i $(i < f)$, satisfying:

$\varphi_{c_i}(\bar{x},\bar{u}) \Rightarrow \theta_{c_i}(\bar{u})$. If some c_i $(i < f)$ is also a normal code for some set agreeing with C_f up to a set of rank $< k$, let $c_f = c_i$. Otherwise, let c be any normal code for a set C' agreeing with C_f up to rank $< k$. Let $\theta'_i(\bar{u})$ be the formula: $\sim (\exists \bar{v})(\mathrm{rk}\{\bar{x}\colon \varphi_{c_i}(\bar{x},\bar{v})\}\triangle\{\bar{x}\colon \varphi_c(\bar{x},\bar{u})\} < k)$. Then $\models \theta'_i(\bar{a})$. Let c_f be the result of modifying c by replacing θ_c by $\theta_{cf} = \theta_c \& \theta'_0 \& \ldots \& \theta'_{f-1}$, and replacing φ_c by $\varphi_c \& \theta_{cf}$. It is easy to see that this construction provides a set of codes satisfying (i) and (ii).

In order to meet (iii), the following modification is needed. For $\sigma \in \mathrm{Sym}(n)$, let $C^\sigma = \{\bar{x}^\sigma\colon \bar{x} \in C\}$, and let c^σ be the code obtained from c in the natural manner so that if $(c,\bar{b})$ is a code for C, then $(c^\sigma,\bar{b})$ is a code for C^σ. ∎

CLAIM: Let $C \subseteq D^n$ be a definable set of rank k. Then there exists a normal code $(c,\bar{b})$ for a set $C(\bar{b})$ agreeing with C up to rank $< k$, such that the family of codes $\{c^\sigma\colon \sigma \in \mathrm{Sym}(n)\}$ satisfies (i).

Proof: The point is that c must have the following property: if $(c,\bar{b})$ and $(c^\sigma,\overline{b'})$ are codes for the same set (up to rank $< k$), then $c = c^\sigma$. Start with any normal code $(c,\bar{b}_0)$ for C (we may assume one exists.) Let B be the set of conjugates of $\overline{b_0}$. Let $G = \{\sigma \in \mathrm{Sym}(n)\colon$ for some $\bar{b}^\sigma \in B$, $C^\sigma(\bar{b}) \sim C(\bar{b}^\sigma)\}$. Clearly G is a subgroup of $\mathrm{Sym}(n)$, $\bar{b}^\sigma$ is defined uniquely for $\bar{b} \in B$ and $\sigma \in G$, and $\bar{b} \mapsto \bar{b}^\sigma$ is a definable action of G on B. By compactness, we may strengthen θ so that if $\models \theta(\bar{b})$ and $\models \theta(\bar{b}')$ and $\sigma \notin G$ then $C^\sigma(\bar{b}) \not\sim C(\bar{b}')$.

We may further strenghten θ so that if $\models \theta(\bar{b})$ and $\sigma \in G$ then there exists $\bar{b}^\sigma$ such that $C^\sigma(\bar{b}) \sim C(\bar{b}^\sigma)$ and $\models \theta(\bar{b}^\sigma)$. Thus the action of G extends to an action on $\{\bar{u}\colon \theta(\bar{u})\}$. Let Q be the quotient of the set by this action, and let θ' be a formula defining the quotient. If $b/G \in Q$, then $\cap_{\sigma\in G} C(\bar{b}^\sigma)$ is definable from $\bar{b}/G$, and agrees with $C(\bar{b})$ up to rank $< k$. Now it is easy to find a code c' with $\theta_{c'} = \theta'$ meeting the claim.

Assume c_i $(i < f)$ have been defined so that c_i is a code for C_i, and (1) holds for $\{c_i^\sigma\colon i < f, \sigma \in \mathrm{Sym}(n)\}$. Find a normal code $(c,\bar{b})$ for C_f satisfying the claim. Now follow the procedure above to find c_f such that (1) holds for $\{c_i^\sigma\colon i \le f, \sigma \in \mathrm{Sym}(n)\}$. ∎

Definition: A **2-code** is a pair (c^1,c^2) of standard codes for L_1, L_2 respectively, such that:

(i) $n(c^1) = n(c^2) = k(c^1) + k(c^2)$.

Let $\bar{x} = \bar{x}^1\hat{\ }\bar{x}^2$ be any partition of the variables $\bar{x}$ into two sets, with $\bar{x}^i$ of length $n^i \geq 1$. Then for some k_1, k_2 with $k_1 + k_2 > n^1$,

(ii) $T_1 \vdash$ for all $\bar{u}$ such that $\theta_{c^1}(\bar{u})$, and all $\overline{x^1}$, $\{\overline{x^2}\colon \varphi_{c^1}(\overline{x^1}\,\overline{x^2}, \bar{u})\}$ has rank $\leq k(c_1) - k_1$.

$T_2 \vdash$ for all $\bar{u}$ such that $\theta_{c^1}(\bar{u})$, and all $\overline{x^1}$, $\{\overline{x^2}\colon \varphi_{c^2}(\overline{x^1}\,\overline{x^2}, \bar{u})\}$ has rank $\leq k(c_2) - k_2$.

Note that $(k(c_1) - k_1) + (k(c_2) - k_2) = n - (k_1 + k_2) < n_2$. If k_1, k_2 are chosen maximal, taking into account (iv)(d) of the definition of a normal code, we have:

(iii) $T_1 \vdash$ for all $\bar{u}$ such that $\theta_{c^1}(\bar{u})$, $\{\overline{x^1}\colon \{\overline{x^2}\colon \varphi_{c^1}(\overline{x^1}\,\overline{x^2}, \bar{u})\}$ has rank $\geq k(c_1) - k_1\}$ has rank k_1.

$T_2 \vdash$ for all $\bar{u}$ such that $\theta_{c^2}(\bar{u})$, $\{\overline{x^1}\colon \{\overline{x^2}\colon \varphi_{c^2}(\overline{x^1}\,\overline{x^2}, \bar{u})\}$ has rank $\geq k(c_2) - k_2\}$ has rank k_2.

(If the rank of $\{\overline{x^1}\colon \{\overline{x^2}\colon \varphi_{c^1}(\overline{x^1}\,\overline{x^2}, \bar{u})\}$ has rank $\geq k(c_1) - k_1\}$ were $>$ or $< k_1$, then the rank of $\{\bar{x}\colon \varphi_{c^1}(\overline{x^1}\,x^2, \bar{u})\}$ would be $>$ or $< k(c_1)$.) We further demand:

(iv) $T_1 \cup T_2 \vdash$ for all $\bar{u}$ such that $\theta_{c^1}(\bar{u})$ and $\theta_{c^2}(\bar{u})$ and all $\bar{x}$ such that $\varphi_{c^1}(\bar{x}, \bar{u})$ and $\varphi_{c^2}(\bar{x}, \bar{u})$, $x_1, \ldots, x_n$ are distinct.

Let $n(c) = n(c^1) = n(c^2)$, $m(c) = \max(m(c^1), m(c^2))$. ∎

LEMMA 3A: *Let c be a 2-code. Let N be an $L_1 \cup L_2$-structure, $N|L_i \models T_i^{\forall}$, M a substructure, $\overline{b^1}, \overline{b^2}$ from M, $A = \{a_1, \ldots, a_n\} \subseteq N, \bar{a} = (a_1, \ldots, a_n)$, $\models \theta_{c^i}(\overline{b^i})$ and $\varphi_{c^i}(\bar{a}, \bar{b}^i)(i = 1, 2)$.*

(a) $d_0(\bar{a}/M) \leq 0$

(b) *If $d_0(\bar{a}/M) = 0$ then $A \subseteq M$ or $A \cap M = \emptyset$*

(c) *If $d_0(\bar{a}/M) = 0$, $A' \subseteq A$, and $d_0(A'/M) \leq 0$, then $A \subseteq M$ or $A' = A$ or $A' = \emptyset$. The same conclusion holds if b^i lies in N^i, where N^i is a model of T_i extending N, and $\bar{b}^i$ is definable over M in N^i.*

Proof:

(a) Let B be a finite set of parameters from M such that $\bar{b}_i$ is definable over B in N^i. Because of our assumption of quantifier elimination for L_i, $d_i(\bar{a}/B) \leq k(c^i)$. So $d_0(\bar{a}/B) = d_1(\bar{a}/B) + d_2(\bar{a}/B) - n \leq 0$.

(b) By (iii) of the definition of a 2-code and the note following (let $\bar{x}^1\hat{\ }\bar{x}^2$ be a partition of the variables corresponding to $A = (A \cap M) \cup (A - M)$).

(c) By (a) and (b) applied to $M \cup A'$ in place of M, either $A = A'$ or $A' = \emptyset$ or $A \subseteq M$ or $d_0'(A/M \cup A') < 0$. In the last case, since also $d_0(A'/M) \leq 0$, we have $d_0(A/M) < 0$, a contradiction. ∎

Let acl_i denote algebraic closure in the sense of T_i.

LEMMA 3B: *Let N be a model of with $T_1^\forall \cup T_2^\forall$ with $\emptyset \leq N$, and $B \subseteq N$. Let $A \subseteq N$ be finite, $d_0(A/B) = 0$, $A \neq \emptyset$, and assume there is no proper nonempty subset A' of A with $d_0(A'/B) < 0$. Let $\bar{a}$ enumerate A. Then there exists a unique 2-code c, $\overline{b^1} \in \mathrm{acl}_1(B)$, and $\overline{b^2} \in \mathrm{acl}_2(B)$ such that $N \models \theta_{c^i}(\bar{b}^i)$ and $\varphi_{c^i}(\bar{a}, \bar{b}^i)(i = 1, 2)$.*

Proof: Let c^1, c^2 be the unique standard codes such that $N \models \theta_{c^i}(\bar{b}^i)$ and $\varphi_{c^i}(\bar{a}, \bar{b}^i)(i = 1, 2)$. A partition of the variables corresponds to a proper nonempty subset A' of A. Let $k_i = d_i(A')$. Then $k_1 + k_2 - n_1 = d_0(A'/B) > 0$. (ii) of the definition of 2-codes follows from (iv)(d) of the definition of normal codes.

THE THEORY T.

DATA. A finite-one integer valued function $\mu*$ defined on 2-codes, satisfying:

(i) $\mu^*(c) \geq m(c) - 1$

(ii) $\mu^*(c) = \mu^*(c')$ if c differs from c' only by a permuation of the variables $x_1, \dots, x_{n(c)}$.

Let $\mu(c) = m(c)n(c) + \mu^*(c)$.

Given a 2-code c and an integer $M \geq m(c)$, let $\Theta_c(\overline{u^1}, \overline{u^2}, \overline{y_1}, \dots, \overline{y_m})$ be the conjunction of the following conditions:

- $\{y_{\nu,l}: 1 \leq \nu \leq M,\ 1 \leq l \leq n(c)\}$ is a set of $M \cdot n(c)$ distinct elements.
- ψ_{c^i} holds of each $m(c^i)$-tuple of y_j's $(i = 1, 2)$
- $f_{c^i}(\bar{y}_1, \dots, \bar{y}_{m(c^i)}) = \bar{u}^i$, and $\theta_{c_i}(\overline{u^i})$ holds $(i = 1, 2)$
- $\varphi_{c^i}(\bar{y}_j, \bar{u}_i)(i = 1, 2, j = 0, ..., M - 1)$.

Note that Θ_c is the conjunction of an L_1-formula and an L_2-formula, $\Theta_c \equiv \Theta_c^1 \& \Theta_c^2$. Let

$$\Theta_c'^i(\bar{y}_1, \dots, \bar{y}_m) = \Theta_c^i(f_{c^1}(\bar{y}_1, \dots, y_{m(c^1)}), f_{c^2}(\bar{y}_1, \dots, y_{m(c^2)}), \bar{y}_1, \dots, \bar{y}_m)$$
$$\Theta_c' = \Theta_c'^1 \& \Theta_c'^2$$

UNIVERSAL AXIOMS.

(i) The universal parts of T_1 and of T_2.

(ii) $\Theta_c'(\bar{y}_1, \dots, \bar{y}_m)$ has no solutions if $M > \mu(c)$.

(iii) Every model N of T satisfies $\emptyset \leq N$.

Explicitly: For each pair $\varphi_1(\bar{x}), \varphi_2(\bar{x})$ of formulas of L_1, L_2 respectively in n variables defining sets of ranks k_1, k_2 respectively with $k_1 + k_2 < n$, the sentence: $\sim (\exists \bar{x})(\varphi_1(\bar{x}) \& \varphi_2(\bar{x}) \& \bigwedge_{i \neq j} x_i \neq x_j)$.

$\forall\exists$ AXIOMS.

(iv) Axioms stating that if $M \models T$, then $M|L_i$ is algebraically closed (as a submodel of a model of T_i).

(v) For each 2-code $c = (c^1, c^2))$ and integer L, an axiom ($\mathrm{v_c}$) stating: For all $\bar{u}^1, \bar{u}^2, W$, if $\theta_{c^i}(\bar{u}^i)$ holds, and W is a set of $n(c)$-tuples of size L, then one of the following:

(a) $(\exists \bar{x})(\bar{x} \notin W$ and $\varphi_{c^1}(\bar{x}, \bar{u}^1)$ and $\varphi_{c^2}(\bar{x}, \bar{u}^2))$,

(b) $(\exists \overline{y_1}, \ldots, \overline{y_r} \in W)\Theta_c(\bar{u}^1, \bar{u}^2, \overline{y_1}, \ldots, \overline{y_r})$ where $r = \mu(c)$.

or

(c) For some 2-code c' and some choice of variables $\bar{y}$ as explained below, $(\exists \bar{y} - \bar{x})(\text{for } i = 1, 2)(\forall * \bar{x} \text{ s.t. } \varphi_{c^i}(\bar{x}, \bar{u}^1))(\Theta'^{i}_{c'}(\bar{y}_0, \ldots, \bar{y}_{\mu(c')}))$
where notation in (c) is as follows:
$y_\nu = (y_{\nu,1}, \ldots, y_{\nu,n(c')})$;
if $\nu \geq m(c')n(c')$, then $y_{\nu,l} = x_j$ (one of the variables in $\bar{x}$);
if $\nu < m(c')n(c')$, then $y_{\nu,l}$ may be either some x_j or a new variable;
$(\exists \bar{y} - \bar{x})$ quantifies out those $y_{i,l}$ that do not appear also as x's;
$(\forall * \bar{x} \ldots)$ means: $\{\bar{x}\colon \ldots\}$ has rank $\geq k(c^i)$.

Recall that $\Theta'^{i}_{c'}$ fails if any two of its arguments are equal. Hence the $y_{\nu,l}$'s may be taken to be distinct variables. Such a choice of $y_{\nu,l}$ from among the x_j's ($\nu \geq m(c')n(c')$) is only possible if $(\mu(c') + 1 - m(c')n(c'))n(c') \leq n(c)$, and in particular $\mu * (c') \leq n(c)/n(c')$. Since $\mu*$ is finite-one, only finitely many 2-codes c' are involved.

In words: c has a solution outside W, unless W contains a "maximal" set of solutions, or for some c', adding a generic solution of c would create too many solutions of c'. The word "maximal" is in quotes since there may well be more than $\mu(c)$ solutions to $\varphi_{c^1}(\bar{x}, \overline{b^1}) \& \varphi_{\overline{c^2}}(\bar{x}, \overline{b^2})$.

Similarly, (a),(c) are not mutually exclusive; (c) only rules out the possibility of a solution generic over W. Also, for given parameters, the axiom has the nature of a disjunction, stating that either c or (some) c' have solutions, but not resolving which.

$\exists\forall$ AXIOMS.

(vi) Axioms ensuring that in a saturated model M of T there exists an infinite d-independent set I.

We will sometimes refer to subtheories of T such as T(i,iii). $T^{\forall}$ means T(i,ii,iii).

LEMMA 4: *Suppose M is a model of $T^{\forall}$ such that:*

(a) *There exists an infinite d-independent $I \subseteq M$.*

(b) *Whenever $M \leq N$, $N \models T^{\forall}$, $\varphi(\bar{x}, \bar{y})$ a quantifier-free formula, $\bar{b}$ from M, and $N \models (\exists \bar{x})\varphi(\bar{x}, \bar{b})$, then $M \models (\exists \bar{x})\varphi(\bar{x}, \bar{b})$.*

Then M is a model of T.

Because of the dimension restriction, (b) may be referred to as "d- existential-closure."

Proof: Assume (a),(b) hold. Only axioms (iv),(v) need to be verified. ∎

CLAIM: Let $n = M \cup \{a\}$ be an $L_1 \cup L_2$-extension of M, $N \models T_1^{\forall} \cup T_2^{\forall}$. Suppose $a \in \mathrm{acl}(M) - M$ in the sense of T_1, and $a \notin \mathrm{acl}(M)$ in the sense of L_2. Then $N \models T^{\forall}$.

Proof: $M \leq N$ is clear. Hence $\emptyset \leq N$, by lemma 1(ii).

Suppose $\Theta'_c(\bar{a}_1, \ldots, \bar{a}_r)$ holds, $r > \mu(c)$. Note that $n(c) > 1$, and exactly one a^i does not lie in M; say $a_{r,n} = a$ and the other $a_{\nu,l}$'s are in M. By lemma 3A applied to $\bar{a}_r, d_0(\bar{a}_r/M) < 0$. This contradicts $M \leq N$. ∎

CLAIM: $M \models T(\text{iv})$.

Proof: Suppose for contradiction that $M|L_1$ is not algebraically closed (for example). Let $N = M \cup \{a\}$ be a model of $T_1^{\forall}$, M a submodel, such that $a \in \mathrm{acl}(M) - M$ in the sense of T_1. Make N into an L_2-structure so that a is an independent generic over M (i.e. $a \notin \mathrm{acl}(M)$). By the previous claim, $M \leq N$ and $N \models T^{\forall}$. Let $\varphi(x, \bar{y})$ be an L_1-formula over M such that $T_1 \models (\forall \bar{y})(\forall x_0 \ldots x_m)($ if $\bigwedge_i \varphi(x_i, \bar{y})$ then $V_{i \neq j} x_i = x_j)$, and $M \models \varphi(a, \bar{b})$, $\bar{b}$ from M. Let S be the (finite) set of solutions of $\varphi(x, \bar{b})$ in M. Then a is a solution of $\varphi(x, \bar{b}) \& x \notin S$. Since M is d-existentially closed, M contains a solution of this formula; a contradiction. ∎

CLAIM: Let $N = M \cup \{a_1, \ldots, a_n\}$, $N \models$ T(i), $M \leq N$. Let c be a 2-code, $\overline{b^1}, \overline{b^2}$ from M, and suppose $N \models \varphi_{c^i}(\bar{a}, \overline{b^1})$. Then one of the following holds:

(a) $N \models T^{\forall}$
(b) There exist $\bar{a}^0, \ldots, \bar{a}^{r_0-1}$ in M, $\Theta_c(\bar{b}^1, \bar{b}^2; \bar{a}^0, \ldots, \bar{a}^{r_0-1})$, $r_0 = \mu(c)$.
(c) There exists a 2-code c', $r_1 \leq m(c')n(c'), \bar{a}^0, \ldots \bar{a}^{r_1-1}$ from N, and $\bar{a}^{r_1}, \ldots, \bar{a}^r$ from $\{a_1, \ldots, a_n\}$, $r = \mu(c')$, such that $\Theta'_{c'}(\bar{a}^0, \ldots, \bar{a}^r)$ holds.

Proof: Suppose (a) fails. $N \models$ T(i) is assumed, and T(iii) follows from this and the assumptions $\emptyset \leq M \leq N$. So T(ii) fails in N: for some 2-code c', $r = \mu(c') + 1$, there are $\bar{a}^0, \ldots, \bar{a}^r$ from N such that $N \models \Theta'_{c'}(\bar{a}^0, \ldots, \bar{a}^r)$. Note that the coordinates of $\bar{a}^0, \ldots, \bar{a}^r$ are distinct. Using the symmetry of Θ', let $\bar{a}^0, \ldots, \bar{a}^{r_0-1}$ be those $\bar{a}_i$'s lying entirely in M, and let $\bar{a}^0, \ldots, \bar{a}^{r_1-1}$ be the $\bar{a}_i$'s with some co-ordinate in M.

Let

$$k(i) = d_0(\bar{a}^i / M \cup \{\bar{a}^0, \ldots, \bar{a}^{i-1}\}).$$

Then by lemma 3A, $k(i) \leq 0$ if $i \geq m(c')$, and $k(i) < 0$ unless $\bar{a}^i \cap M = \emptyset$ or $\bar{a}^i \subseteq M$. So $k(i) < 0$ if $\max(r_0, m(c')) \leq i < r_1$. ∎

CASE: $r_0 \geq m(c')$
If $r_0 = r$ then all $\bar{a}_i$'s are in M, contradicting $M \models T^{\forall}$. So $\bar{a}^{r_0}$ exists and is not entirely in M. Now $d_0(\bar{a}^{r_0}/M) = k(r_0) \leq 0$; since $M \leq N$, $d(\bar{a}/M) = 0$; by lemma 3A, this implies $\bar{a}^{r_0} = \bar{a}^{\sigma}$ for some permutation σ of the variables. By the definition of standard codes, and the uniqueness part of lemma 3B, $c' = c^{\sigma}$, $r_0 = r - 1 = \mu(c') = \mu(c^{\sigma}) = \mu(c)$. So (b) holds.

CASE: $r_0 < m(c')$.
$\Sigma_{i<r_1} k(i) = d_0(\bar{a}^0, \ldots, \bar{a}^{r_1-1}/M) \geq 0$.

For $i < m(c')$, $k(i) \leq (n(c') - 1)$ (since $\bar{a}^i$ has $n(c')$ coordinates, at least one of which is in M).

For $m(c') \leq i < r_1$, $k(i) < 0$. Thus $\Sigma_{i<r_1} k(i) \leq (m(c')n(c') - 1) - (r_1 - m(c'))$. $0 \leq m(c')(n(c') - 1) - (r_1 - m(c'))$, or $r_1 \leq m(c')n(c')$. This gives (c).

CLAIM: $M \models$ T(v)

Proof: Let c be a normal code, $\bar{b}^1, b^2$ from M, $\models \theta_{c^i}(\bar{b}^i)$. Let W be a finite set of $n(c)$-tuples from M. Let $\{a_1, \ldots, a_n\}$ be new elements, $N = M \cup \{a_1, \ldots, a_n\}$; make N into an L_i-extension of M in such a way that $N \models \varphi_{c^i}(\bar{a}\bar{b}^i)$ and $d_i(\bar{a}/M) = k(c_i)$. So $d_0(\bar{a}/M) = n - k(c_1) - k(c_2) = 0$. By lemma 3A, $M \leq N$. So one of the possibilities of the previous claim applies. If (a) holds, then in $N \varphi_{c_1}(\bar{x}\bar{b}^1) \& \varphi_{c_2}(\bar{x}, \bar{b}^2)$ has a solution outside W, so as M is d-existentially closed this must also be true in M. If (b) holds, there are two cases: either one of the $\bar{a}^i$'s is outside W (so the first alternative of axiom $\mathrm{v_c}$ holds again) or they are all in W, so the second alternative holds. If (c) is the case, let $y_{\nu,l} = x_j$ where the l'th co-ordinate of $\bar{a}^\nu$ is a_j. Then (c) of the axiom is immediately verified.

LEMMA 5: *Let M be a model of T(i-iv). Let $\varphi_i(\bar{x}, \bar{a}_i)$ be an L_i-formula over M in n variables of rank k_i, and let J be a set of pairwise disjoint solutions of $\varphi_1 \& \varphi_2$ in M. If $k_1 + k_2 \leq n$, then J is finite.*

Proof: By induction on n. Suppose on the contrary that J is infinite. Replacing J by a subset (twice), we may assume that J forms a Morley sequence in $M|L_i$ over a finite set $A \supseteq \overline{a^i} (i = 1,2)$. Let $k^i = d_i(\bar{c}/A)(\bar{c} \in J)$. Then $k^i \leq k_i$. Let J_m be a subset of J of size m. Then $d_i(J_m/A) = mk^i$, while $A \cup \bigcup J_m$ has size $|A| + mn$. (The latter uses the fact that the tuples enumerate pairwise disjoint sets.) So $d_0(A \cup \bigcup J_m/A) = m(k^1 + k^2 - n) \geq -d_0(A)$. For large m this implies $k^1 + k^2 - n \geq 0$. Hence $k^1 + k^2 \geq n \geq k_1 + k_2$. So $k^i = k_i$ and $k_1 + k_2 = n$. Thus $d_0(\bar{c}/A) = 0$ for $c \in J$.

If there is some proper sub-tuple $\overline{c'}$ of $\bar{c}$ with $d_0(\overline{c'}/A) = 0$ we get a contradiction to the induction hypothesis. Otherwise, by lemma 3B, there exists a 2-code c and $\bar{b}^i \in \mathrm{acl}_i(A)$ (in M) such that $M \models \theta_{c^i}(\bar{b}^i)$ and $\varphi_{c^i}(\bar{c}, \bar{b}^i)$. Thus by the definition of a normal code, $\models \psi_{c^i}(\bar{c}^1, \ldots, \bar{c}^m)$ for any $\bar{c}^1, \ldots, \bar{c}^m$ from J ($m = m(c^i)$). Now the fact that J is infinite contradicts axiom (ii). ∎

LEMMA 5′: *Let M be a model of T(i-iv), let c be a 2-code, and let $\bar{a}^1, \bar{a}^2$ be tuples from M such that $M \models \theta_{c^i}(\bar{a}^i)$. Then there are only finitely many $n(c)$-tuples $\bar{x}$ in M such that $\models \varphi_{c^i}(\bar{x}, \bar{a}^i)(i = 1,2)$.*

Proof: Suppose on the contrary that J is an infinite set of such tuples. Again we may assume that J forms a Morley sequence in $M|L_i$ over a finite set $A \supseteq \overline{a^i}$ ($i = 1,2$). Let A^i be the set of co-ordinates of the i'th tuple in J. We may assume that the A^i's form a Δ-system, i.e. that $A^i \cap A^j = A_0$ for $i \neq j$. Let $A = A_0 \cup \{\bar{a}^1, \bar{a}^2\}$. By the previous lemma, the A^i's cannot be pairwise disjoint, so $A^i \cap A \neq \emptyset$. Since there A^i's are distinct, $A^i \not\subseteq (A \cup A^0 \cup \cdots \cup A^{i-1})$. By lemma 3A, $d_0(A^i/A \cup A^0 \cup \cdots \cup A^{i-1}) < 0$. Thus $d_0(A \cup A_0 \cup \cdots \cup A^n) < 0$ for large n, a contradiction. ∎

LEMMA 6: *Let B_1, B_2 be submodels of models M_1, M_2 of T such that $B_i \leq M_i$ and $d(M_i/B_i) = 0$. Let $f\colon B_1 \to B_2$ be a bijection preserving the atomic relations of $L_1 \cup L_2$. Then f extends to an isomorphism $M_1 \to M_2$.*

Proof: By symmetry and exhaustion, it suffices to show that if $B_1 \neq M$ then f can be properly extended to another map meeting the same conditions. ∎

CLAIM: f extends to an atomic isomorphism of $\mathrm{acl}_1(B_1)$ with $\mathrm{acl}_1(B_2)$. Moreover, $\mathrm{acl}_1(B_i) \leq M_i$.

Proof: If $E \subseteq \mathrm{acl}_1(B_i) - B_i$, $\mathrm{card}(E) = n \geq 1$, and $d_2(E/B_i) < n$, then for any sufficiently large $F \subseteq B_i$ we find that $d_0(E/F) < 0$, contradicting the assumption $B_i \leq M_i$. Thus $\mathrm{acl}_1(B_i) - B_i$ is a set of independent generic elements over B_i in $M_i|L_2$. Thus any bijection of $\mathrm{acl}_1(B_1)$ with $\mathrm{acl}_1(B_2)$ extending f is an L_2-isomorphism. So any extension of f to an L_1-isomorphism of $\mathrm{acl}_1(B_1)$ with $\mathrm{acl}_1(B_2)$ will satisfy the requirement. The fact that $\mathrm{acl}_1(B_i) \leq M_i$ follows from $B_i \leq M_i$ together with the observation that $d_0(E/B_i) = 0$ for $E \subseteq \mathrm{acl}_1(B_i)$.

We are reduced to the case that B_i is algebraically closed in the sense of L_1, and similarly in the sense of L_2. We may assume $B_1 = B_2 = B$ and f is the identity. Let $a_0 \in M_1 - B$. Then $d(a_0/B_0) = 0$ for some finite $B_0 \subseteq B$. So for some $A = \{a_1, \ldots, a_n\} \subseteq M_1 - B$ and some finite $B_0 \leq B$, $d_0(A/B_0) = 0$. Choose A with $n \geq 1$ least possible.

Let $k_i = d_i(\bar{a}/B)$. By lemma 3B, there exists a 2-code c with $n(c) = n$, $k(c^i) = k_i$, and $\bar{e}^1, \bar{e}^2 \in B$, such that $M_1 \models \varphi_{c^i}(\bar{a}, e^i)$.

Let $W = \{\bar{x} \in B^n\colon \models \varphi_{c^i}(\bar{x}, \bar{e}^i),\ i = 1,2\}$. By lemma 5′, W is finite. By axiom (v) in M_2 one of the following cases occurs.

CASE A: There exists $\overline{a'} \in M_2^n - W$ such that $\models \varphi_{c^i}(\overline{a'}, \bar{e}^i), i = 1,2$.

In this case extend f to $B \cup \{\bar{a}\}$ by mapping $\bar{a}$ to $\overline{a'}$. Since $\overline{a'} \in W$,$\overline{a'}$ does not entirely lie inside B. As $B \leq M_2$, $d_0(\overline{a'}/B) \geq 0$, so by lemma 3A $d_0(\overline{a'}/B) = 0$ and $\overline{a'}$ lies entirely outside B. Thus $d_1(\overline{a'}/B) + d_2(\overline{a'}/B) = n$. Since $k_i \geq d_i(\overline{a'}/B)$ and $k_1 + k_2 = n$, $d_i(\overline{a'}/B) = k_i$. Now $\varphi_{c^i}(\bar{x}, \bar{e}_i)$ determines a unique type of rank k_i; so the L_i-type of $\overline{a'}$ over B equals that of $\overline{a'}$ over B. So f is indeed an $L_1 \cup L_2$-embedding. The fact that $B \cup \{\bar{a}\} \leq M_1$ and $B \cup \{\overline{a'}\} \leq M_2$ again follows from $d_0(\bar{a}/B) = 0$, $d_0(\overline{a'}/B) = 0$.

CASE B: There are $\bar{a}_1, \ldots, \bar{a}_r$ in W, $r = \mu(c)$, $\Theta_c(\bar{e}_1, \bar{e}_2, \bar{a}_1, \ldots, \bar{a}_r)$.

By part (e) of the definition of a normal code, since $\bar{a}$ is a generic realization (from the point of view of $M_1|L_i$) of the set coded by $(c^i, \bar{e}_i)$, we also have $\Theta_c(\bar{e}_1, \bar{e}_2, \bar{a}_1, \ldots, \bar{a}_r, \bar{a})$. This contradicts axiom (ii) in M_1.

CASE C: Let $\overline{a'}$ realize $tp(\bar{a}/\bar{b}_i)$, $\overline{a'} \perp M_2|\bar{b}_i$, $A' = \{a'_1, \ldots, a'_n\}$, $M' = M \cup A'$ (a model of $T_1^{\forall} \cup T_2^{\forall}$). Then for some 2-code c' and some $\bar{a}_0, \ldots, \bar{a}_{\mu(c')}$ from $M_2 \cup A'$, $\Theta'_{c'}(\bar{a}_0, \ldots, \bar{a}_{\mu(c')})$ holds. Moreover $\bar{a}_i \in {A'}^{n(c')}$ for $i \geq m(c')n(c')$. Let $r = \mu(c')$, $r_1 = r - m(c') + 1$. As $\mu * (c') \geq m(c') - 1$, $r_1 \geq m(c')n(c')$. Let $\bar{d}^i = f_{c'^i}(\bar{a}_{r_1}, \ldots, \bar{a}_r)$ (the last $m(c')$ $\bar{a}_i$'s.) Then $\varphi_{c'^i}(\bar{a}_j, \bar{d}^i)$ holds for $j = 0, \ldots, \mu(c')$. By lemma 3A applied to c', $d_0(\bar{a}_j/A') \leq 0$. ∎

CLAIM: Each $\bar{a}_j$ lies in $B \cup A'$.

Proof: Fix j and suppose $\bar{a}_j$ does not lie in $B \cup A'$. Let $\bar{c}_j$ be the part of $\bar{a}_j$ outside $B \cup A'$. Then $\bar{c}_j \perp A'|B$ in the sense of both L_1 and L_2. As $B \leq M_2$, $d_0(\bar{c}_j/B) \geq 0$; so $d_0(\bar{c}_j/B \cup A') \geq 0$. Eqivalently, $d_0(\bar{a}_j/B \cup A') \geq 0$. By lemma 3A again, $d_0(\bar{a}_j/B \cup A') = 0$ and $\bar{a}_j = \bar{c}_j$. It follows that $\bar{a}_j$ is a generic solution over $B \cup A'$ of the L_i-set coded by $(c'^i, \bar{d}^i)$. Since $\bar{a}_j \perp A'|B$ and $\bar{d}^i$ is the canonical base of $\mathrm{tp}(\bar{a}_j/B \cup A')$, $\bar{d}^i$ must be L_i-algebraic over B. As B is L_i-algebraically closed, $\bar{d}^i$ is from B. But then $d_0(\bar{a}_r/B) = 0$. By 3A(c), it follows that $\bar{a}_r$ enumerates all of A', so $\mu(c') = r = 0$, a contradiction.

Thus the atomic type of $B \cup A'$ contradicts axiom (ii) for c'. This is a contradiction since the same type is realized in M_1 (by $B \cup A$). So case (c) is not possible. ∎

COROLLARY: Let B_1, B_2 be submodels of models M_1, M_2 of T such that $B_i \leq M_i$. Let f: $B_1 \to B_2$ be a bijection preserving the atomic relations of $L_1 \cup L_2$. Then f is a partial elementary map.

Proof: First observe that if $\bar{M}_i$ is an elemtary extension of M_i, then $M_i \leq \bar{M}_i$. Indeed if $A \subseteq \bar{M}_i - M_i$ is finite and $d_0(A/B) < 0$ where $B \subseteq M_i$ is finite and $A \cap M_i \subseteq B$, then there exists $A' \subseteq M_i$ with $d_0(A'/B) < 0$. If we choose $B = cl_0(B)$ in M, this is not possible.

This allows us to assume that the dimension of M_1 over B_1 in the sense of d-dependence equals that M_2 over B_2. Let J_i be a d-basis for M_i over B_i, and extend f to a bijection $\bar{f}$: $B_1 \cup J_1 \to B_2 \cup J_2$ arbitrarily. The hypotheses of lemma 6 are easily verified, and so $\bar{f}$ extends to an isomorphism of M_1 with M_2. Hence f is elementary. ∎

COROLLARY: Let M be a model of T. The dependence relation described in lemma 1(viii) coincides with algebraic closure in M.

Proof: In one direction, we must show that if $d(a/B) = 0$ then $a \in \mathrm{acl}(B)$. We may assume B is finite. So $d(a/B) = 0$ means $d_0(cl_0(\{a\} \cup B)) = d_0(cl_0(B))$. As acl is transitive, it suffices to show:

(a) $cl_0(B) \subseteq \mathrm{acl}(B)$

(b) If $B = cl_0(B)$, $C = B \cup \{a_1, \ldots, a_n\}$, $C = cl_0(C)$, $d_0(C/B) = 0$, and there is no proper subset of C properly containing B with the last two properties, then $C \subseteq \mathrm{acl}(B)$

(a) follows from lemmas 3B and 5'. (b) requires only axioms (i) and (iii), and is left to the reader.

Conversely, suppose $d(a/B) \neq 0$; we want to show that $a \notin \mathrm{acl}(B)$. We may assume $B = cl_0(B)$. Let N be a model of T such that $B \leq N$ and there exists an infinite set $I \subseteq N$, d-independent over B. By the previous corollary, for each $c \in I$ there exists an elementary embedding of M into N with $a \mapsto c$. Thus $a \notin \mathrm{acl}(B)$. ∎

COROLLARY: T is complete, consistent and strongly minimal.

Proof: Consistency follows from lemma 4: let M_0 be an infinite independent set for D_i as an L_i-structure. Then it is clear that M_0 satisfies $T^\forall$. Let M be a d-existentially closed model of $T^\forall$ extending M_0. Then $M \models T$.

By the previous corollary and lemma 6 we have the hypotheses of the following claim, which therefore finishes the proof. ∎

CLAIM: Let T be a theory. Assume:

(a) Algebraic closure gives a dependence relation on any model of M.

(b) Any bijection between transcendence bases of models of T extends to an isomorphism of the models.

Then T is complete and strongly minimal.

Proof: If T',T'' are two completions of T, one can find models M',M'' of the dimension $|T|$; then by (b) $M' \approx M''$ so $T' = T''$. For strong minimality, let A be a finite subset of a model of T, and let E be an A-definable equivalence relation on elements; we must show that all but one class is finite. If this fails, choose A as small as possible. If $A' \subset A \subset \mathrm{acl}(A')$, consider the intersection of all conjugates of E over A'; it too must have more than one infinite class. Thus A is independent. Pick independent over A elements a, b, c with a, b in one infinite E-class and c in another; by (b) the map fixing A and a and exchanging b, c is elementary; a contradiction. This shows A is strongly minimal. ∎

References

[B] John Baldwin, *α_T is finite for $\aleph_1$-categorical T*, Trans. Amer. Math. Soc. **181** (1973), 37–51.

[D] Lau Van den Dries, *Model theory of fields: decidability, and bounds for polynomial ideals*, Doctoral Thesis, Univ. Utrecht, 1978.

[HL] E. Hrushovski and J. Loveys, *Structure of strongly minimal modules*, in preparation.

[H] E. Hrushovski, *A strongly minimal set*, to appear.

[HP] E. Hrushovski and A. Pillay, *Weakly normal groups*, Logic Colloquium 85, North-Holland, Amsterdam, 1986.

[L] Serge Lang, *Introduction to Algebraic Geometry*, Interscience, New York, 1964.

[M] Dave Marker, *Semi-algebraic expansions of C*, Trans. Amer. Math. Soc. **320** (1989), 581–592.

[P1] Bruno Poizat, *Missionary mathematics*, J. Symb. Logic **53** (1988), 137–145.

[P2] Bruno Poizat, *Une théorie de Galois imaginaire*, J. Symb. Logic **48** (1983), 1151–1170.

[Pi] A. Pillay, *An Introduction to Stability Theory*, Oxford University Press, 1983.

[S] A. Seidenberg, *Constructions in algebra*, Trans. Amer. Math. Soc. **197** (1974), 273–313.